2019-Sep-26 (pre-lecture) MATB44-2019 Prof. Yun William Yu

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Outline:

- Proof of Banach fixed pt theorem
- Multidimensional Picard iteration
- Statement of the Picard-Lindelöf Thm

Last time:

- We gave many examples of contractions, including Jaccard iteration.
- To finish the existence-uniqueness proof for ODE solutions, we need to prove the Banach fixed pt theorem, describe multidimensional Picard iteration, and show that the last is a contraction.

Theorem (Teschl 2.1): Banach fixed-point theorem (contractive principle)

Let C be a (nonempty) closed subset of a Banach space X and let $K: C \to C$ be a contraction. Then K has a unique fixed point $\bar{x} \in C$ s.t.

$$\|K^n(x) - \bar{x}\| \le \frac{\theta^n}{1-\theta}\|K(x) - x\|, \quad x \in C.$$

proof. **Existence** Fix $x_0 \in C$ and consider the sequence

$$x_n = K^n(x_0).$$

Then we have
$$\begin{aligned}\|x_{n+1} - x_n\| &\le \theta\|x_n - x_{n-1}\| \\ &\le \theta^2\|x_{n-1} - x_{n-2}\| \\ &\vdots \\ &\le \theta^n\|x_1 - x_0\|.\end{aligned}$$

By the triangle inequality (since we have a metric), for $n > m$

$$\begin{aligned}\|x_n - x_m\| &= \|(x_n - x_{n-1}) + (x_{n-1} - x_{n-2}) + \cdots (x_{m+1} - x_m)\| \\ &\le \|x_n - x_{n-1}\| + \cdots + \|x_{m+1} - x_m\| \\ &\le \theta^{n-1}\|x_1 - x_0\| + \cdots + \theta^m\|x_1 - x_0\| \\ &= \theta^m \sum_{j=0}^{n-m-1} \theta^j \|x_1 - x_0\|\end{aligned}$$

$$= \Theta^m \sum_{j=0}^{n-m-1} \Theta^j \|x_1 - x_0\|$$

$$= \Theta^m \cdot \frac{1-\Theta^{n-m}}{1-\Theta} \|x_1 - x_0\| \le \frac{\Theta^m}{1-\Theta} \|x_1 - x_0\|.$$

Because for any ε, we can find an M s.t. $\forall\, n, m \ge M$,

$\|x_n - x_m\| \le \varepsilon$, this sequence is Cauchy.

Because we are in a Banach space, Cauchy sequences tend to some limit, which here we call $\bar{x}$.

Note, $\|K(\bar{x}) - \bar{x}\| = \lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$, so $\bar{x}$ is a fixed point.

Uniqueness: Let $\bar{x}$ and $\bar{y}$ be fixed points of K, $\bar{x}, \bar{y} \in C$.

Then $\|K(\bar{x}) - K(\bar{y})\| = \|\bar{x} - \bar{y}\|$.

But $\|K(\bar{x}) - K(\bar{y})\| \le \Theta \|\bar{x} - \bar{y}\|$ because K is a contraction.

So $\|\bar{x} - \bar{y}\| \le \Theta \|\bar{x} - \bar{y}\|$.

$\Theta < 1$, so this can only be true if $\|\bar{x} - \bar{y}\| = 0$

$\Rightarrow \bar{x} = \bar{y}$. $\square$

Picard iteration on systems of ODEs

Last time: $\dot{x} = f(t, x)$, $x(t_0) = x_0$, $x : \mathbb{R} \to \mathbb{R}$.

Let $K : C(U, \mathbb{R}) \to C(U, \mathbb{R})$, $U \subseteq \mathbb{R}^2$ an open set,

$$K(x)(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds.$$

Then the sequence

$$x_0 = x_0$$
$$x_1 = K(x_0)$$
$$x_2 = K(x_1)$$

of Picard iterates converges to the solution $x(t)$.

This time: $\dot{x} = f(t, x)$, $x(t) = x_0$, $x : \mathbb{R} \to \mathbb{R}^n$

Let $K : C(U, \mathbb{R}) \to C(U, \mathbb{R})$, $U \subseteq \mathbb{R}^{n+1}$ an open set.

Let $K : C(U, \mathbb{R}) \longrightarrow C(U, \mathbb{R})$, $U \subseteq \mathbb{R}^{n+1}$ an open set.

$$K(x)(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds.$$

Then the sequence $x_0(t) = x_0$, $x_m(t) = K(x_{m-1})$ converges to the solution $x(t)$.

All we did was replace a scalar valued x with a vector-valued function.
Or equivalently, a system of first-order ODEs.

Ex. Say we have a first-order system of 2 ODEs.

I am going to use **bold green** for vectors to keep things clear

Let $\quad \mathbf{x}(t) = \begin{pmatrix} t x_2 \\ x_1 x_2 \end{pmatrix}$, $\quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\mathbf{x}_1(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} \int_0^t t\, dt \\ \int_0^t 1\, dt \end{pmatrix} = \begin{pmatrix} 1 + \frac{t^2}{2} \\ 1 + t \end{pmatrix}$$

$$\mathbf{x}_2(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} \int_0^t t(1+t)\, dt \\ \int_0^t \left(1 + \frac{t^2}{2}\right)(1+t)\, dt \end{pmatrix}$$

$$= \begin{pmatrix} 1 + \frac{t^2}{2} + \frac{t^3}{3} \\ 1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{8} \end{pmatrix}$$

$$\mathbf{x}_3(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} \int_0^t t\left(1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{8}\right) dt \\ \int_0^t \left(1 + \frac{t^2}{2} + \frac{t^3}{3}\right)\left(1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{8}\right) dt \end{pmatrix}$$

Or equivalently,

$$\frac{dy}{dt} = tz, \qquad \frac{dz}{dt} = yz$$

$$y(0) = 1, \qquad z(0) = 1.$$

$$y_1(t) = 1 + \int_0^t t\, dt = 1 + \frac{t^2}{2} \qquad z_1(t) = 1 + \int_0^t 1\, dt = 1 + t$$

$$y_2(t) = 1 + \int_0^t t(1+t)\, dt = 1 + \frac{t^2}{2} + \frac{t^3}{3}$$

$$z_2(t) = 1 + \int_0^t \left(1 + \frac{t^2}{2}\right)(1+t)\, dt = 1 + \int_0^t \left(1 + t + \frac{t^2}{2} + \frac{t^3}{2}\right) dt = 1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{8}$$

$$y_3(t) = 1 + \int_0^t t\left(1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{8}\right) dt$$

$$z_3(t) = 1 + \int_0^t \left(1 + \frac{t^2}{2} + \frac{t^3}{3}\right)\left(1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{8}\right) dt$$

See Tenenbaum 57.35 for full arithmetic.

Notice: Picard iteration also works on systems of first-order ODEs. Recall that we can convert any system of higher-order ODEs into a system of first-order ODEs.

into a system of first-order ODEs.

So, all we have to do now is prove when Picard iteration is a contraction, leading to the most important theorem of MATB44:

Theorem (Picard-Lindelöf):
(Teschl 2.2)

Suppose $f: C(U, \mathbb{R}^n)$, where U is an open subset of $\mathbb{R}^{n+1}$, and $(t_0, x_0) \in U$. If f is locally Lipschitz continuous in the 2nd argument, uniformly with respect to the 1st argument, then there exists a unique local solution $\bar{x}(t) \in C^1(I)$ of the initial value problem, where I is some interval around t_0.

Define: A function $f: X \to Y$ is **continuous** if for every $x \in X$, $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. for every $y \in X$ with $\|x-y\| < \delta$, $\|f(x) - f(y)\| < \varepsilon$.

Define: A function $f: X \to Y$ is **uniformly continuous** if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. for every $x, y \in X$ with $\|x-y\| < \delta$, $\|f(x) - f(y)\| < \varepsilon$.

Define: A function $f: X \to Y$ is **Lipschitz continuous** if $\exists L \geq 0$ s.t. for every $x, y \in X$, $\|f(x) - f(y)\| \leq L \|x-y\|$.

Remark: Continuous implies no jumps.

Uniformly continuous implies bounds on the rate of change depend only on the distance between two points, rather than the specific points

Lipschitz continuous implies the rate of change is strictly bounded everywhere. (or at least locally everywhere in a neighborhood)

On a compact set (e.g. closed interval),

Continuously differentiable

$\Rightarrow$ Lipschitz continuous

(Result from Analysis)

(Analysis)

Lipschitz continuous

Uniformly continuous $\Rightarrow$ continuous

So the theorem is more generally applicable, but we can just think of $f(t,x)$ as having continuous derivative, because in that special case, all the uniformities we care about are satisfied.